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Research Article

Stability of uncertain *2-D* discrete delayed systems with saturation

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Abstract
Based on the Roesser model with saturation nonlinearities (SNL) and time-varying delays (TVDs), this paper studies the global asymptotic stability (GAS) of two-dimensional (2-D) uncertain discrete systems (DSs). The underlying system involves norm-bounded parameter uncertainties. By utilizing the idea of Wirtinger-based inequality (WBI) with reciprocal convex inequality (RCI), a new criterion is derived to ensure the GAS of 2-D systems. Numerical examples demonstrate the advantages of the proposed method. With the MATLAB software and YALMIP 3.0, it is found that the obtained criterion provides less stringent results than an existing criterion.

Keywords
2-D discrete system, Reciprocal convex inequality, Saturation nonlinearity, Time-varying delay, Wirtinger-based inequality.

1.Introduction
Two-dimensional (2-*D*) discrete systems (DSs) have been extensively used in control systems, optical fibre networks, river pollution modelling, geophysics, speech processing, etc. [1–4]. As a result, over the last few decades, there has been a lot of interest in studying the stability properties of such systems. In the hardware realization of DSs on finite word length machines (e.g., microcontrollers, special-purpose digital hardware and digital signal processers, etc.), overflow and quantization nonlinearities are inevitable. Such nonlinearities in DSs may result in oscillations and instability [5–8]. The commonly employed overflow correction schemes in DSs are 2's complement, saturation, triangular and zeroing. Many problems have been studied in the literature to examine the global asymptotic stability (GAS) of 2-*D* DSs with overflow [9–13].

Another source of system instability and poor performance is time delay. It may frequently occur in realistic systems owing to transmission delays, measurement delays, computational delays, etc. The presence of delay in practical systems may cause undesired transient responses and, thus, lead the system towards instability [9, 13–22].

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As a result, investigating the GAS of 2-*D* DSs employing time-varying delays (TVDs) is an important research topic. Several authors have paid attention to the problem of GAS in 2-*D* delayed DSs [15–17, 19–21].

Parameter uncertainties may emerge in practical systems as a result of changes in system parameters, modelling errors, or other issues that are ignored. These uncertainties may cause the implemented system to become unstable. Several works on the stability of 2-*D* DSs with parameter uncertainties have been reported [15, 17, 19, 21].

While obtaining delay-dependent stability criteria for DSs, several methods (e.g., free weighting matrix (FWM) method, inequality-based method, etc.) have been accounted for in handling the sum terms present in the forward difference of the Lyapunov-Krasovskii function (LKF). The FWM method makes the criteria more complex by introducing free matrix variables into the system analysis. The sum term is solved by employing well-known inequalities (such as reciprocal convex inequality (RCI) [23], Jensen inequality (JI) [24] and Wirtinger-based inequality (WBI) [25]) in the inequality-based method. The RCI is a lower bounded lemma for a linear combination of positive functions, where the coefficients are the inverses of convex parameters. This lemma may aid in the reduction of decision variables as well as conservatism [23, 25]. The criteria achieved by employing the WBI are generally less stringent than those achieved via JI [26]. Nevertheless, there is still an enormous scope to improve the results reported in [23, 26–28].

The task of designing 2-*D* DSs preserving the GAS is an appealing problem. The GAS problem of 2-*D* Roesser model [29] has been studied [2, 9, 11, 17, 20]. The stability criteria of 2-*D* DSs in the Fornasini-Marchesini second local state space (FMSLSS) model have been reported in [30]. In [9, 31], the stability problem of 2-*D* delayed systems using saturation nonlinearities (SNL) has been explored. The stability problem of uncertain 2-*D* delayed systems with SNL has not been investigated so far, to the best of our awareness.

1. For 2-*D* DSs under the combined influence of TVDs, uncertainties and SNL, the linear matrix inequality (LMI) based stability criterion is provided.
2. To estimate the sum terms in the forward difference of the LKF, the WBI technique is combined with RCI.
3. Further, a new stability criterion is proposed in corollary, which yields more relaxed outcomes than the previous criterion [9].

The paper is prepared as follows. Section 1 gives the introduction, a description of the system and some relevant preliminaries. The previous studies in the literature that are related to the present study are reviewed in section 2. A new GAS criterion for 2-*D* DSs under the combined influence of TVDs, uncertainties and SNL is presented in section 3. Further, a new criterion is provided as a corollary that does not take into account the influence of uncertainty in 2-*D* DSs. Section 4 demonstrates the superiority of the obtained results by using two examples. Section 5 deals with the discussion of the proposed results. Finally, a conclusion is provided in section 6.

Notations: In this paper, $\boldsymbol{A} < \mathbf{0}$ $(> \mathbf{0})$ stands for $\boldsymbol{A}$ is a symmetric negative definite (positive definite) matrix; $\boldsymbol{A} \geq \mathbf{0}$ indicates that $\boldsymbol{A}$ is a symmetric positive semidefinite matrix; $\boldsymbol{A}_1 \oplus \boldsymbol{A}_2$ refers to the matrix $\begin{bmatrix} \boldsymbol{A}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_2 \end{bmatrix}$; $\mathbb{R}^p$ denotes the p-dimensional Euclidean space; $\mathbb{R}^{p \times q}$ is the real set of $p \times q$ matrices; $\mathbf{0}$ and $\boldsymbol{I}$ are the null and identity matrices of appropriate dimension, respectively; $\sup\{\cdot\}$ is the supremum of a set; $\|\cdot\|$ is any vector or matrix norm; $\lfloor \gamma \rceil$ stands for the nearest integer to γ; *diag* $\{\eth_1, \eth_2, ..., \eth_n\}$ is a diagonal matrix in which $\eth_1, \eth_2, ..., \eth_n$ are diagonal elements; Z_+ is the set of non-negative integers; $*$ refers to the symmetric terms in a symmetric matrix.

1.1 System description

This paper considers the 2-*D* DSs (based on the Roesser model [29]) with uncertainties, TVDs and SNL. The underlying system is given by Equations 1a-1e:

$$\boldsymbol{\aleph}_{11}(\mu,\ \nu) = \begin{bmatrix} \boldsymbol{\aleph}^h(\mu+1,\ \nu) \\ \hdashline \boldsymbol{\aleph}^v(\mu,\ \nu+1) \end{bmatrix} = \boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)) = \begin{bmatrix} \boldsymbol{f}^h(\boldsymbol{\sigma}^h(\mu,\ \nu)) \\ \hdashline \boldsymbol{f}^v(\boldsymbol{\sigma}^v(\mu,\ \nu)) \end{bmatrix}, \tag{1a}$$

$$\boldsymbol{f}^h(\boldsymbol{\sigma}^h(\mu,\ \nu)) = \begin{bmatrix} f_1^h(\sigma_1^h(\mu,\ \nu)) & f_2^h(\sigma_2^h(\mu,\ \nu)) & \cdots & f_m^h(\sigma_m^h(\mu,\ \nu)) \end{bmatrix}^{\mathrm{T}}, \tag{1b}$$

$$\boldsymbol{f}^v(\boldsymbol{\sigma}^v(\mu,\ \nu)) = \begin{bmatrix} f_1^v(\sigma_1^v(\mu,\ \nu)) & f_2^v(\sigma_2^v(\mu,\ \nu)) & \cdots & f_n^v(\sigma_n^v(\mu,\ \nu)) \end{bmatrix}^{\mathrm{T}}, \tag{1c}$$

$$\boldsymbol{\sigma}(\mu,\ \nu) = \begin{bmatrix} \boldsymbol{\sigma}^h(\mu,\ \nu) \\ \hdashline \boldsymbol{\sigma}^v(\mu,\ \nu) \end{bmatrix} = (\boldsymbol{X} + \Delta\boldsymbol{X}) \begin{bmatrix} \boldsymbol{\aleph}^h(\mu,\ \nu) \\ \hdashline \boldsymbol{\aleph}^v(\mu,\ \nu) \end{bmatrix} + (\boldsymbol{X}_d + \Delta\boldsymbol{X}_d) \begin{bmatrix} \boldsymbol{\aleph}^h(\mu - d^h(\mu),\ \nu) \\ \hdashline \boldsymbol{\aleph}^v(\mu,\ \nu - d^v(\nu)) \end{bmatrix}, \tag{1d}$$

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix},\ \boldsymbol{X}_d = \begin{bmatrix} \boldsymbol{X}_{d_{11}} & \boldsymbol{X}_{d_{12}} \\ \boldsymbol{X}_{d_{21}} & \boldsymbol{X}_{d_{22}} \end{bmatrix},\ \Delta\boldsymbol{X} = \begin{bmatrix} \Delta\boldsymbol{X}_{11} & \Delta\boldsymbol{X}_{12} \\ \Delta\boldsymbol{X}_{21} & \Delta\boldsymbol{X}_{22} \end{bmatrix},\ \Delta\boldsymbol{X}_d = \begin{bmatrix} \Delta\boldsymbol{X}_{d_{11}} & \Delta\boldsymbol{X}_{d_{12}} \\ \Delta\boldsymbol{X}_{d_{21}} & \Delta\boldsymbol{X}_{d_{22}} \end{bmatrix}, \tag{1e}$$

where $\mu \in Z_+$ and $\nu \in Z_+$ are the horizontal and vertical spatial coordinates, respectively. $\aleph^h(\mu,\nu) \in \mathbb{R}^m$ and $\aleph^v(\mu,\nu) \in \mathbb{R}^n$ are the state vectors in horizontal direction (HD) and vertical direction (VD), respectively. The $\boldsymbol{X_{11}}, \boldsymbol{X_{d_{11}}} \in \mathbb{R}^{m\times m}$, $\boldsymbol{X_{12}}, \boldsymbol{X_{d_{12}}} \in \mathbb{R}^{m\times n}$, $X_{21}, \boldsymbol{X_{d_{21}}} \in \mathbb{R}^{n\times m}$ and $\boldsymbol{X_{22}}, \boldsymbol{X_{d_{22}}} \in \mathbb{R}^{n\times n}$ are the known coefficient matrices. The $\boldsymbol{\Delta X_{11}}, \boldsymbol{\Delta X_{d_{11}}} \in \mathbb{R}^{m\times m}$, $\boldsymbol{\Delta X_{12}}, \boldsymbol{\Delta X_{d_{12}}} \in \mathbb{R}^{m\times n}$, $\Delta X_{21}, \boldsymbol{\Delta X_{d_{21}}} \in \mathbb{R}^{n\times m}$ and $\boldsymbol{\Delta X_{22}}, \boldsymbol{\Delta X_{d_{22}}} \in \mathbb{R}^{n\times n}$ are the unknown matrices and $\boldsymbol{f}(\cdot)$ represents the SNL. The $d^h(\mu)$ and $d^v(\nu)$ are TVDs in HD and VD, respectively [9, 17]. The TVDs are assumed to satisfy Equation 2.

$$0 < d_1^h \leqslant d^h(\mu) \leqslant d_2^h, \quad 0 < d_1^v \leqslant d^v(\nu) \leqslant d_2^v, \tag{2}$$

Where

- d_1^h lower delay bound along HD,
- d_2^h upper delay bound along HD,
- d_1^v lower delay bound along VD,
- d_2^v upper delay bound along VD.

The $f_i^h(\sigma_i^h(\mu,\nu))$ and $f_i^v(\sigma_i^v(\mu,\nu))$ representing the SNL along HD and VD, are given by Equation 3a and Equation 3b, respectively.

$$f_i^h(\sigma_i^h(\mu,\nu)) = \begin{cases} \sigma_i^h(\mu,\nu), & |\sigma_i^h(\mu,\nu)| \leqslant 1, \\ 1, & \sigma_i^h(\mu,\nu) > 1, \\ -1, & \sigma_i^h(\mu,\nu) < -1, \end{cases}$$
$$i = 1, 2, \ldots, m, \tag{3a}$$

$$f_i^v(\sigma_i^v(\mu,\nu)) = \begin{cases} \sigma_i^v(\mu,\nu), & |\sigma_i^v(\mu,\nu)| \leqslant 1, \\ 1, & \sigma_i^v(\mu,\nu) > 1, \\ -1, & \sigma_i^v(\mu,\nu) < -1, \end{cases}$$
$$i = 1, 2, \ldots, n. \tag{3b}$$

The norm-bounded uncertainties are assumed in the form of Equations 4a-4c:

$$\boldsymbol{\Delta X} = \boldsymbol{P}_0 \mathsf{T}_0 \boldsymbol{Q}_0, \quad \boldsymbol{\Delta X}_d = \boldsymbol{P}_1 \mathsf{T}_1 \boldsymbol{Q}_1, \tag{4a}$$

Where

$$\boldsymbol{P}_i = \begin{bmatrix} \boldsymbol{P}_i^h \\ \boldsymbol{P}_i^v \end{bmatrix}, \boldsymbol{P}_i^h \in \mathbb{R}^{m\times p_i}, \boldsymbol{P}_i^v \in \mathbb{R}^{n\times p_i}, \quad i = 0, 1, \tag{4b}$$

$$\boldsymbol{Q}_i = [\boldsymbol{Q}_i^h \quad \boldsymbol{Q}_i^v], \boldsymbol{Q}_i^h \in \mathbb{R}^{q_i\times m}, \ \boldsymbol{Q}_i^v \in \mathbb{R}^{q_i\times n}, \quad i = 0, 1 \tag{4c}$$

are known. The unknown matrix $\mathsf{T}_i (i = 0, 1)$ satisfies Equation 4d.

$$\mathsf{T}_i^T \mathsf{T}_i \leq I, \quad i = 0, 1. \tag{4d}$$

The system is assumed to have a finite set of boundary conditions [9], i.e., there are scalars $M > 0$ and $N > 0$ satisfying Equation 5.

$$\begin{aligned} \aleph^h(\mu,\nu) &= \boldsymbol{\rho}_{\mu\nu}, \quad \forall\ 0 \leqslant \nu < M, \quad -d_2^h \leqslant \mu \leqslant 0, \\ &= \boldsymbol{0}, \quad \forall\ \nu \geq M, \quad -d_2^h \leqslant \mu \leqslant 0, \\ \aleph^v(\mu,\nu) &= \boldsymbol{\varrho}_{\mu\nu}, \quad \forall\ 0 \leqslant \mu < N, \quad -d_2^v \leqslant \nu \leqslant 0, \\ &= \boldsymbol{0}, \quad \forall\ \mu \geq N, \quad -d_2^v \leqslant \nu \leqslant 0, \\ \boldsymbol{\rho}_{00} &= \tilde{\mathbf{n}}_{00}. \end{aligned} \tag{5}$$

It is worth noting that Equations 1-5 cover a broad range of 2-*D* practical uncertain systems with TVDs and SNL. Such systems cover 2-*D* control systems [32] with SNL [16], 2-*D* DSs employed on a digital signal processor [33], wireless sensor networks [34], vehicle control systems [35], networked control systems [36], etc.

The purpose of this work is to examine the GAS of 2-*D* DS given by Equations 1-5 and to derive a more relaxed criterion by utilizing WBI along with RCI.

1.2Preliminaries

In this subsection, we recall the following definition and lemmas.

Definition [9]**:** The system in Equation 1 is globally asymptotically stable if $\lim_{\ell\to\infty} \partial_\ell = 0$ for all boundary conditions belonging to Equation 5, where ∂_ℓ is represented by Equation 6.

$$\partial_\ell = \sup\left\{ \left\| \begin{bmatrix} \aleph^h(\mu,\nu) \\ \aleph^v(\mu,\nu) \end{bmatrix} \right\| : \mu + \nu = \ell, \quad \mu, \nu \geq 1 \right\}. \tag{6}$$

Lemma 1 [25]**:** Let a matrix $\boldsymbol{T} > \boldsymbol{0}$ and a, b, μ are integers fulfilling $\mu \geq b \geq a \geq 0$, if $\boldsymbol{\chi}(\mu, a, b)$ is represented as Equation 7a

$$\boldsymbol{\chi}(\mu, a, b) = \begin{cases} \dfrac{1}{b-a}\left[\left(2\displaystyle\sum_{s=\mu-b}^{\mu-a-1} \aleph(s)\right) + \aleph(\mu-a) - \aleph(\mu-b)\right], & a < b, \\ 2\aleph(\mu-a), & a = b, \end{cases} \tag{7a}$$

then the inequality given in Equations 7b-7e holds true.

$$-(b-a)\sum_{s=\mu-b}^{\mu-a-1} \boldsymbol{\kappa}^{\mathrm{T}}(s)\boldsymbol{T}\boldsymbol{\kappa}(s) \leq -\begin{bmatrix} \boldsymbol{\Omega}_0 \\ \boldsymbol{\Omega}_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{T} & \boldsymbol{0} \\ \boldsymbol{0} & 3\boldsymbol{T} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}_0 \\ \boldsymbol{\Omega}_1 \end{bmatrix}, \tag{7b}$$

Where

$$\boldsymbol{\kappa}(s) = \aleph(s+1) - \aleph(s), \tag{7c}$$

$$\boldsymbol{\Omega}_0 = \aleph(\mu-a) - \aleph(\mu-b), \tag{7d}$$

$$\boldsymbol{\Omega}_1 = \aleph(\mu-a) + \aleph(\mu-b) - \boldsymbol{\chi}(\mu, a, b). \tag{7e}$$

Here, $\boldsymbol{\Omega}_0$ is the mean value gained by $\aleph(\mu)$ over [a, b]. $\boldsymbol{\Omega}_1$ is the difference between the mean value of $\aleph(\mu)$ and its average over [a, b].

Lemma 2 [23]**:** For any vectors $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, matrices $\boldsymbol{R}$, $\boldsymbol{N}$ and non-negative scalars α_1, α_2 satisfying Equation 8a and Equation 8b,

$$\alpha_1+\alpha_2=1,\ \begin{bmatrix}\boldsymbol{R} & \boldsymbol{N}\\ * & \boldsymbol{R}\end{bmatrix}\geq \boldsymbol{0}, \tag{8a}$$

$$\boldsymbol{\beta}_i=\boldsymbol{0} \text{ if } \alpha_i=0 \quad (i=1,\ 2) \tag{8b}$$

then the inequality shown in Equation 8c is satisfied.

$$-\frac{1}{\alpha_1}\boldsymbol{\beta}_1^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\beta}_1-\frac{1}{\alpha_2}\boldsymbol{\beta}_2^{\mathrm{T}}\boldsymbol{R}\boldsymbol{\beta}_2\leq -\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\boldsymbol{R} & \boldsymbol{N}\\ * & \boldsymbol{R}\end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{bmatrix}. \tag{8c}$$

Lemma 3 [37]**:** Suppose $\Pi, \Omega, \top$ and $\boldsymbol{\Theta}$ be real matrices with $\boldsymbol{\Theta}$ fulfilling $\boldsymbol{\Theta}=\boldsymbol{\Theta}^{\mathrm{T}}$. Then, the inequality given by Equation 9a holds true:

$$\Theta+\Pi\top\Omega+\Omega^T\top^T\Pi^T<0, \tag{9a}$$

$\forall\ \top^T\top\leq I$ iff there exists a scalar $\in>0$ satisfying Equation 9b.

$$\boldsymbol{\Theta}+\in^{-1}\boldsymbol{\Pi}\ \boldsymbol{\Pi}^{\mathrm{T}}+\in\boldsymbol{\Omega}^{\mathrm{T}}\boldsymbol{\Omega}<\boldsymbol{0}. \tag{9b}$$

2.Literature review

In the last few decades, 2-*D* systems have been widely used in many research fields. In [38], the stability of 2-*D* DS in FMSLSS model subject to exogenous nonlinear disturbances has been considered. The design of robust controller for 2-*D* Markovian jump systems with uncertainties has been discussed in [39]. Stability analysis of systems with TVDs and generalized overflow nonlinearities has been studied in [40]. An investigation of local control schemes for 2-*D* nonlinear DSs in Roesser model has been done in [41]. The problem of GAS of 2-*D* digital filters based on the FMSLSS model with saturation has been investigated in [11]. Stability conditions for 2-*D* linear shift invariant DSs has been reported in [42]. The problems of establishing L_2 gain and structural stability of the mixed 2-*D* continuous-DSs have been discussed in [43]. The problem of ensuring stability for mixed 2-*D* continuous-DSs has been addressed in [44]. The structural stability of the continuous-discrete fractional-order 2-*D* Roesser model is investigated in [45]. The exponential stability of continuous descriptor systems based on 2-*D* Roesser model has been examined in [46]. The $\boldsymbol{H}_\infty$ stability of 2-*D* Roesser-like continuous delayed systems has been investigated in [47]. Using Takagi-Sugeno fuzzy models, [48] has investigated the controller design problem for 2-*D* nonlinear systems.

In the existing literature, the sum terms in the forward difference of the LKF have been handled using a variety of ways. The FWM technique generally yields computationally complex stability criterion. In the inequality-based technique, the sum terms in the forward difference of the LKF are generally handled by applying well-known inequalities, like the RCI method [23], the JI method [24] and the WBI method [25]. The criteria developed using the WBI approach are generally more relaxed than those obtained through the JI method [26].

The stability issues of 2-*D* DSs with TVDs have been investigated in [15−17]. The stability problem of 2-*D* delayed fuzzy systems given by the Roesser model has been considered in [49]. In [50], the stability problem for a class of 2-*D* DSs based on Roesser model with actuator saturation and TVDs has been discussed. The stability of delayed continuous-discrete systems has been analysed in [51].

From the literature survey mentioned above, it is apparent that the stability analysis of DSs with uncertainties, FWN, and delays is an imperative and challenging problem. Although several results are presented in this area, there are still copious possibilities for improvement in terms of computational burden and/or conservativeness.

3.Methods

The following theorem provides a new methodology for testing GAS of the system given by Equations 1-5.

Theorem 1: For given integers d_i^h, d_i^v $(i=1,2)$, fulfilling $d_2^h>d_1^h>0$ and $d_2^v>d_1^v>0$, the GAS of the system represented by Equations 1-5 is assured if there exist matrices

$$\boldsymbol{0}<\boldsymbol{W^h}=\begin{bmatrix}\boldsymbol{W}_1^h & \boldsymbol{W}_2^h & \boldsymbol{W}_3^h\\ * & \boldsymbol{W}_4^h & \boldsymbol{W}_5^h\\ * & * & \boldsymbol{W}_6^h\end{bmatrix}\in\mathbb{R}^{3m\times 3m},$$

$$\boldsymbol{0}<\boldsymbol{W^v}=\begin{bmatrix}\boldsymbol{W}_1^v & \boldsymbol{W}_2^v & \boldsymbol{W}_3^v\\ * & \boldsymbol{W}_4^v & \boldsymbol{W}_5^v\\ * & * & \boldsymbol{W}_6^v\end{bmatrix}\in\mathbb{R}^{3n\times 3n},\quad \boldsymbol{0}<\boldsymbol{H}_i^h\in\mathbb{R}^{m\times m}\ (i=1,2,3),$$

$\boldsymbol{0}<\boldsymbol{H}_i^v\in\mathbb{R}^{n\times n}\ (i=1,2,3)$,
$\boldsymbol{0}<\boldsymbol{T}_i^h\in\mathbb{R}^{m\times m}\ (i=1,2)$,
$\boldsymbol{0}<\boldsymbol{T}_i^v\in\mathbb{R}^{n\times n}\ (i=1,2)$,

$\boldsymbol{Y}^h = \begin{bmatrix} \boldsymbol{Y}_{11}^h & \boldsymbol{Y}_{12}^h \\ \boldsymbol{Y}_{21}^h & \boldsymbol{Y}_{22}^h \end{bmatrix} \in \mathbb{R}^{2m\times 2m}$,
$\boldsymbol{Y}^v = \begin{bmatrix} \boldsymbol{Y}_{11}^v & \boldsymbol{Y}_{12}^v \\ \boldsymbol{Y}_{21}^v & \boldsymbol{Y}_{22}^v \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$ and scalars
$0 < \epsilon_0, 0 < \epsilon_1, 0 < \alpha_{ij}, 0 < \beta_{ij}$
$(i,\ j = 1, 2, ..., m+n,\quad i \neq j)$ such that the inequalities shown in Equations 10a-11d hold true:

$$\begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{Y}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \geq \mathbf{0}, \tag{10a}$$

$$\begin{bmatrix} \boldsymbol{T}^v & \boldsymbol{Y}^v \\ * & \boldsymbol{T}^v \end{bmatrix} \geq \mathbf{0}, \tag{10b}$$

$$\varnothing(d^h(\mu) = d_1^h,\ d^v(\nu) = d_1^v) < \mathbf{0}, \tag{11a}$$

$$\varnothing(d^h(\mu) = d_2^h,\ d^v(\nu) = d_1^v) < \mathbf{0}, \tag{11b}$$

$$\varnothing(d^h(\mu) = d_1^h,\ d^v(\nu) = d_2^v) < \mathbf{0}, \tag{11c}$$

$$\varnothing(d^h(\mu) = d_2^h,\ d^v(\nu) = d_2^v) < \mathbf{0}, \tag{11d}$$

where $\varnothing(d^h(\mu),\ d^v(\nu))$ is given by Equation 12a

$$\varnothing(d^h(\mu),\ d^v(\nu)) = \left[\begin{array}{cccccccccc}
\varnothing_{11} + \in_0 \boldsymbol{Q}_0^{\mathrm{T}}\boldsymbol{Q}_0 & \mathbf{0} & ((\boldsymbol{W}_3 - \boldsymbol{W}_2)/2) - 2\boldsymbol{T}_1 & -\boldsymbol{W}_3/2 & \varnothing_{15} + 3T_1 & \varnothing_{16} & \varnothing_{17} & \varnothing_{18} + (W_2^T/2) + X^T C & 0 & 0 \\
* & \varnothing_{22} + \in_1 \boldsymbol{Q}_1^{\mathrm{T}}\boldsymbol{Q}_1 & \varnothing_{23} & \varnothing_{24} & 0 & \varnothing_{26} & \varnothing_{27} & X_d^T C & 0 & 0 \\
* & * & -\boldsymbol{H}_1 - 4(\boldsymbol{T}_1 + \boldsymbol{T}_2) & \varnothing_{34} & \varnothing_{35} + 3T_1 & \varnothing_{36} + 3T_2 & \varnothing_{37} & (W_3^T - W_2^T)/2 & 0 & 0 \\
* & * & * & -\boldsymbol{H}_2 - 4\boldsymbol{T}_2 & -\Upsilon_1(W_5^T/2) & \varnothing_{46} & \varnothing_{47} & -W_3^T/2 & 0 & 0 \\
* & * & * & * & -3T_1 & 0 & 0 & \Upsilon_1(W_2^T/2) & 0 & 0 \\
* & * & * & * & * & -3T_2 & -Y_{22} & \Upsilon_2(W_3^T/2) & 0 & 0 \\
* & * & * & * & * & * & -3T_2 & \Upsilon_3(W_3^T/2) & 0 & 0 \\
* & * & * & * & * & * & * & W_1 - \varnothing_{18} - (C + C^T) & C^T P_0 & C^T P_1 \\
* & * & * & * & * & * & * & * & -\in_0 I & 0 \\
* & * & * & * & * & * & * & * & * & -\in_1 I
\end{array}\right] \tag{12a}$$

and the parameters in Equation 12a are given by Equations 12b-12w:

$$\varnothing_{11} = -\boldsymbol{W}_1 + ((\boldsymbol{W}_2 + \boldsymbol{W}_2^{\mathrm{T}})/2) - 4\boldsymbol{T}_1 + \sum_{i=1}^{3}\boldsymbol{H}_i + \Upsilon_4\boldsymbol{H}_3 - \varnothing_{18}, \tag{12b}$$

$$\varnothing_{15} = \Upsilon_1(\boldsymbol{W}_4 - \boldsymbol{W}_2)/2, \tag{12c}$$

$$\varnothing_{16} = \Upsilon_2(\boldsymbol{W}_5 - \boldsymbol{W}_3)/2, \tag{12d}$$

$$\varnothing_{17} = \Upsilon_3(\boldsymbol{W}_5 - \boldsymbol{W}_3)/2, \tag{12e}$$

$$\varnothing_{18} = -(\Upsilon_1^2\boldsymbol{T}_1 + \Upsilon_4^2\boldsymbol{T}_2), \tag{12f}$$

$$\varnothing_{22} = -\boldsymbol{H}_3 - 8\boldsymbol{T}_2 + \boldsymbol{Y}_{11} + \boldsymbol{Y}_{11}^{\mathrm{T}} + \boldsymbol{Y}_{12} + \boldsymbol{Y}_{12}^{\mathrm{T}} - \boldsymbol{Y}_{21} - \boldsymbol{Y}_{21}^{\mathrm{T}} - \boldsymbol{Y}_{22} - \boldsymbol{Y}_{22}^{\mathrm{T}}, \tag{12g}$$

$$\varnothing_{23} = -2\boldsymbol{T}_2 - \boldsymbol{Y}_{11}^{\mathrm{T}} - \boldsymbol{Y}_{12}^{\mathrm{T}} - \boldsymbol{Y}_{21}^{\mathrm{T}} - \boldsymbol{Y}_{22}^{\mathrm{T}}, \tag{12h}$$

$$\varnothing_{24} = -2\boldsymbol{T}_2 - \boldsymbol{Y}_{11} + \boldsymbol{Y}_{12} + \boldsymbol{Y}_{21} - \boldsymbol{Y}_{22}, \tag{12i}$$

$$\varnothing_{26} = 3\boldsymbol{T}_2 + \boldsymbol{Y}_{21}^{\mathrm{T}} + \boldsymbol{Y}_{22}^{\mathrm{T}}, \tag{12j}$$

$$\varnothing_{27} = 3\boldsymbol{T}_2 - \boldsymbol{Y}_{21} + \boldsymbol{Y}_{22}, \tag{12k}$$

$$\varnothing_{34} = \boldsymbol{Y}_{11} - \boldsymbol{Y}_{12} + \boldsymbol{Y}_{21} - \boldsymbol{Y}_{22}, \tag{12l}$$

$$\varnothing_{35} = \Upsilon_1(\boldsymbol{W}_5^{\mathrm{T}} - \boldsymbol{W}_4)/2, \tag{12m}$$

$$\varnothing_{36} = \Upsilon_2(\boldsymbol{W}_6 - \boldsymbol{W}_5)/2, \tag{12n}$$

$$\varnothing_{37} = (\Upsilon_3(\boldsymbol{W}_6 - \boldsymbol{W}_5)/2) + \boldsymbol{Y}_{12} + \boldsymbol{Y}_{22}, \tag{12o}$$

$$\varnothing_{46} = -\Upsilon_2(\boldsymbol{W}_6/2) - \boldsymbol{Y}_{21}^{\mathrm{T}} + \boldsymbol{Y}_{22}^{\mathrm{T}}, \tag{12p}$$

$$\varnothing_{47} = -\Upsilon_3(\boldsymbol{W}_6/2) + 3\boldsymbol{T}_2, \tag{12q}$$

$$\Upsilon_1 = diag\{d_1^h, d_1^v\}, \tag{12r}$$

$$\Upsilon_2 = diag\{d^h(\mu) - d_1^h, d^v(\nu) - d_1^v\}, \tag{12s}$$

$$\Upsilon_3 = diag\{d_2^h - d^h(\mu), d_2^v - d^v(\nu)\}, \tag{12t}$$

$$\Upsilon_4 = diag\{d_{12}^h, d_{12}^v\}, \tag{12u}$$

$$\boldsymbol{T}^h = \boldsymbol{T}_2^h \oplus 3\boldsymbol{T}_2^h,\ \boldsymbol{T}^v = \boldsymbol{T}_2^v \oplus 3\boldsymbol{T}_2^v, \tag{12v}$$

$$d_{12}^h = d_2^h - d_1^h,\ d_{12}^v = d_2^v - d_1^v \tag{12w}$$

and

$\boldsymbol{W}_i = \boldsymbol{W}_i^h \oplus \boldsymbol{W}_i^v\ (i = 1,\ 2,\ ...,\ 6),\ \boldsymbol{H}_i = \boldsymbol{H}_i^h \oplus \boldsymbol{H}_i^v\ (i = 1,\ 2, 3),$
$\boldsymbol{T}_i = \boldsymbol{T}_i^h \oplus \boldsymbol{T}_i^v\ (i = 1,\ 2),\quad \boldsymbol{Y}_{11} = \boldsymbol{Y}_{11}^h \oplus \boldsymbol{Y}_{11}^v,$

$\boldsymbol{Y}_{12} = \boldsymbol{Y}_{12}^{h} \oplus \boldsymbol{Y}_{12}^{v}$, $\boldsymbol{Y}_{21} = \boldsymbol{Y}_{21}^{h} \oplus \boldsymbol{Y}_{21}^{v}$, $\boldsymbol{Y}_{22} = \boldsymbol{Y}_{22}^{h} \oplus \boldsymbol{Y}_{22}^{v}$. Here, $\boldsymbol{C} = [c_{i\,j}] \in \mathbb{R}^{(m+n)\times(m+n)}$ denotes a matrix defined by Equations 13a-13c [10].

$$c_{i\,i} = \sum_{j=1,\, j\neq i}^{(m+n)} (\alpha_{i\,j} + \beta_{i\,j}), \qquad i = 1,\ 2,\ ...,\ m+n, \tag{13a}$$

$$c_{i\,j} = \alpha_{i\,j} - \beta_{i\,j}, \qquad i,\ j = 1,\ 2,\ ...,\ m+n \quad (i \neq j), \tag{13b}$$

$$\alpha_{i\,j} > 0,\ \ \beta_{i\,j} > 0, \qquad i,\ j = 1,\ 2,\ ...,\ m+n \quad (i \neq j). \tag{13c}$$

Appendix III contains the proof of Theorem 1.

Remark 1: Theorem 1 can be used to determine the GAS of a DS described by Equations 1-5. By selecting a suitable LKF along with the bounding techniques (WBI and RCI), delay-dependent LMI-based stability conditions (Equations 10a-11d) are derived. The conditions in Theorem 1 are computationally tractable. The permitted delay range is generally used to determine how conservative a delay-dependent stability criterion is. To achieve less stringent stability results, the delay ranges in HD and VD should be as large as possible so that the GAS of the DS is ensured over the allowable delay ranges.

Figure 1 depicts the flow chart for the proposed technique for a given 2-*D* DS. This flow chart accepts the system parameters (i.e., $\boldsymbol{X}$, $\boldsymbol{X}_d$, $\boldsymbol{P}_0$, $\boldsymbol{P}_1$, $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, d_1^h, d_2^h, d_1^v, d_2^v of a given 2-*D* DS described by Equations 1-5) as input. Then, the validity of the GAS conditions in Theorem 1 are examined over the delay ranges $d_1^h \leq d^h(\mu) \leq d_2^h$ $d_1^v \leq d^v(\nu) \leq d_2^v$ using MATLAB along with YALMIP 3.0 [37, 52]. If Theorem 1 leads to feasible solution for the considered system, the system becomes globally asymptotically stable over the given delay ranges. If Theorem 1 fails to provide feasible solution, no conclusion on the GAS can be drawn. Though *Figure 1* shows a flow chart for 2-*D* DSs with uncertainties, a flow chart for 2-*D* DSs without uncertainties can easily be conceived.

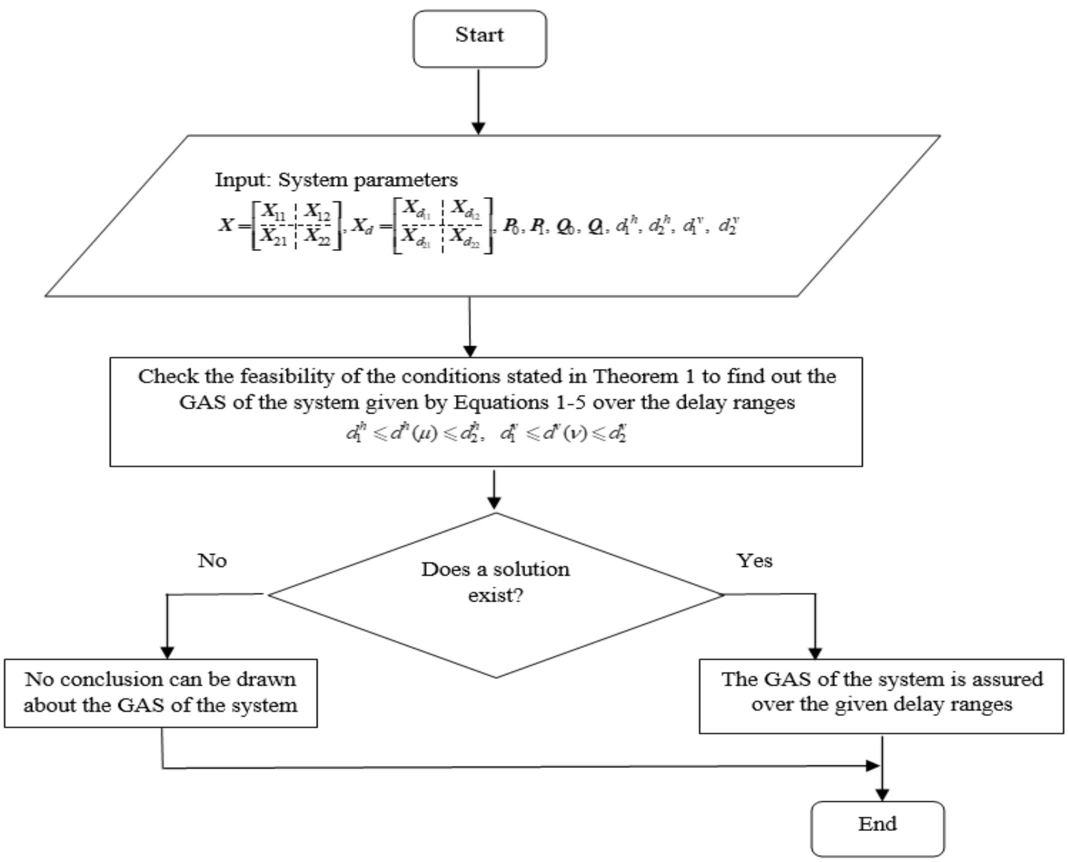


Figure 1 Flow chart for the proposed method

Remark 2: The lessened conservatism obtained in this work is mainly due to the utilization of WBI with RCI, which bounds $\boldsymbol{S}_2^h(\mu,\ \nu)$ as shown in Equation 14: $\boldsymbol{S}_2^h(\mu,\ \nu)\ \le$

$$-\begin{bmatrix} \aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu) \\ \aleph^h(\mu-d_1^h,\ \nu)+\aleph^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \\ \aleph^h(\mu-d^h(\mu),\ \nu)-\aleph^h(\mu-d_2^h,\ \nu) \\ \aleph^h(\mu-d^h(\mu),\ \nu)+\aleph^h(\mu-d_2^h,\ \nu)-\boldsymbol{\chi}(\mu,\ d^h(\mu),\ d_2^h) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{Y}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \begin{bmatrix} \aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu) \\ \aleph^h(\mu-d_1^h,\ \nu)+\aleph^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,\ d^h(\mu)) \\ \aleph^h(\mu-d^h(\mu),\ \nu)-\aleph^h(\mu-d_2^h,\ \nu) \\ \aleph^h(\mu-d^h(\mu),\ \nu)+\aleph^h(\mu-d_2^h,\ \nu)-\boldsymbol{\chi}(\mu,\ d^h(\mu),\ d_2^h) \end{bmatrix} \quad (14)$$

with $\begin{bmatrix} \boldsymbol{T}^h & \boldsymbol{Y}^h \\ * & \boldsymbol{T}^h \end{bmatrix} \ge \mathbf{0}$. By considering $\boldsymbol{Y}^h=\mathbf{0}$ and $d^h(\mu)=d_2^h$, Equation 14 reduces to Equation 15.

$$\boldsymbol{S}_2^h(\mu,\ \nu)\le-\begin{bmatrix} \aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu) \\ \aleph^h(\mu-d_1^h,\ \nu)+\aleph^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,\ d_1^h,d^h(\mu)) \end{bmatrix}^{\mathrm{T}} \boldsymbol{T}^h \begin{bmatrix} \aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu) \\ \aleph^h(\mu-d_1^h,\ \nu)+\aleph^h(\mu-d^h(\mu),\ \nu)-\boldsymbol{\chi}(\mu,d_1^h,\ d^h(\mu)) \end{bmatrix}. \quad (15)$$

By following Remark 6 [26] and Remark 1 [53], we have Equation 16:

$$\boldsymbol{S}_2^h(\mu,\ \nu)\le-\left[\aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu)\right]^{\mathrm{T}} \boldsymbol{T}_2^h\left[\aleph^h(\mu-d_1^h,\ \nu)-\aleph^h(\mu-d^h(\mu),\ \nu)\right] \quad (16)$$

which is identical to the condition obtained via JI. Thus, it may be concluded that Equation 15 is less conservative than that obtained by JI.

Now, we have the following corollary as an outcome of Theorem 1.

Corollary 1: Consider the 2-*D* DS (Equations 1-5) without uncertainties ($\Delta\boldsymbol{X}=\Delta\boldsymbol{X}_d=\mathbf{0}$), i.e., the system transforms to Equation 17.

$$\aleph_{11}(\mu,\ \nu)=\boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu)), \quad (17a)$$

$$\boldsymbol{\sigma}(\mu,\ \nu)=\ \boldsymbol{X}\begin{bmatrix} \aleph^h(\mu,\ \nu) \\ \hline \aleph^v(\mu,\ \nu) \end{bmatrix}+\boldsymbol{X}_d\begin{bmatrix} \aleph^h(\mu-d^h(\mu),\ \nu) \\ \hline \aleph^v(\mu,\ \nu-d^v(\nu)) \end{bmatrix}. \quad (17b)$$

Then, the system in Equation 17 is globally asymptotically stable if there exist

$$\begin{bmatrix} \boldsymbol{W}_1^h & \boldsymbol{W}_2^h & \boldsymbol{W}_3^h \\ * & \boldsymbol{W}_4^h & \boldsymbol{W}_5^h \\ * & * & \boldsymbol{W}_6^h \end{bmatrix}\in\mathbb{R}^{3m\times 3m}$$

$$\mathbf{0}<\boldsymbol{W}^v=\begin{bmatrix} \boldsymbol{W}_1^v & \boldsymbol{W}_2^v & \boldsymbol{W}_3^v \\ * & \boldsymbol{W}_4^v & \boldsymbol{W}_5^v \\ * & * & \boldsymbol{W}_6^v \end{bmatrix}\in\mathbb{R}^{3n\times 3n},$$

$\mathbf{0}<\boldsymbol{H}_i^h\in\mathbb{R}^{m\times m}\ (i=1,2,3)$,
$\mathbf{0}<\boldsymbol{H}_i^v\in\mathbb{R}^{n\times n}\ (i=1,2,3)$,
$\mathbf{0}<\boldsymbol{T}_i^h\in\mathbb{R}^{m\times m}\ (i=1,2)$,
$\mathbf{0}<\boldsymbol{T}_i^v\in\mathbb{R}^{n\times n}\ (i=1,2)$,
$\boldsymbol{Y}^h=\begin{bmatrix} \boldsymbol{Y}_{11}^h & \boldsymbol{Y}_{12}^h \\ \boldsymbol{Y}_{21}^h & \boldsymbol{Y}_{22}^h \end{bmatrix}\in\mathbb{R}^{2m\times 2m}$,
$\boldsymbol{Y}^v=\begin{bmatrix} \boldsymbol{Y}_{11}^v & \boldsymbol{Y}_{12}^v \\ \boldsymbol{Y}_{21}^v & \boldsymbol{Y}_{22}^v \end{bmatrix}\in\mathbb{R}^{2n\times 2n}$ and scalars
$0<\alpha_{ij},\ 0<\beta_{ij}\ (i,\ j=1,2,...,m+n,\quad i\ne j)$

satisfying Equation 10a, Equation 10b along with Equations 18a-18d:

$$\bar{\varnothing}(d^h(\mu)=d_1^h,\ d^v(\nu)=d_1^v)<\ \mathbf{0}, \quad (18a)$$

$$\bar{\varnothing}(d^h(\mu)=d_2^h,\ d^v(\nu)=d_1^v)<\ \mathbf{0}, \quad (18b)$$

$$\bar{\varnothing}(d^h(\mu)=d_1^h,\ d^v(\nu)=d_2^v)<\ \mathbf{0}, \quad (18c)$$

$$\bar{\varnothing}(d^h(\mu)=d_2^h,\ d^v(\nu)=d_2^v)<\ \mathbf{0}, \quad (18d)$$

where $\bar{\varnothing}(d^h(\mu),\ d^v(\nu))$ is given by Equation 19.

$\bar{\varnothing}(d^h(\mu),\ d^v(\nu))=$

$$\begin{bmatrix} \varnothing_{11} & \mathbf{0} & ((\boldsymbol{W}_3-\boldsymbol{W}_2)/2)-2\boldsymbol{T}_1 & -\boldsymbol{W}_3/2 & \varnothing_{15}+3\boldsymbol{T}_1 & \varnothing_{16} & \varnothing_{17} & \varnothing_{18}+(\boldsymbol{W}_2^{\mathrm{T}}/2)+\boldsymbol{X}^{\mathrm{T}}\boldsymbol{C} \\ * & \varnothing_{22} & \varnothing_{23} & \varnothing_{24} & \mathbf{0} & \varnothing_{26} & \varnothing_{27} & \boldsymbol{X}_d^{\mathrm{T}}\boldsymbol{C} \\ * & * & -\boldsymbol{H}_1-4(\boldsymbol{T}_1+\boldsymbol{T}_2) & \varnothing_{34} & \varnothing_{35}+3\boldsymbol{T}_1 & \varnothing_{36}+3\boldsymbol{T}_2 & \varnothing_{37} & (\boldsymbol{W}_3^{\mathrm{T}}-\boldsymbol{W}_2^{\mathrm{T}})/2 \\ * & * & * & -\boldsymbol{H}_2-4\boldsymbol{T}_2 & -\Upsilon_1(\boldsymbol{W}_5^{\mathrm{T}}/2) & \varnothing_{46} & \varnothing_{47} & -\boldsymbol{W}_3^{\mathrm{T}}/2 \\ * & * & * & * & -3\boldsymbol{T}_1 & \mathbf{0} & \mathbf{0} & \Upsilon_1(\boldsymbol{W}_2^{\mathrm{T}}/2) \\ * & * & * & * & * & -3\boldsymbol{T}_2 & -\boldsymbol{Y}_{22} & \Upsilon_2(\boldsymbol{W}_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & -3\boldsymbol{T}_2 & \Upsilon_3(\boldsymbol{W}_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & * & \boldsymbol{W}_1-\varnothing_{18}-(\boldsymbol{C}+\boldsymbol{C}^{\mathrm{T}}) \end{bmatrix}. \quad (19)$$

The proof of Corollary 1 is shown in *Appendix I.*

4.Result

Two examples are considered to exemplify the importance of the main findings.

Example 1: Consider the 2-*D* DS described by Equations 1-5 along with Equations 20a-20d.

$$\boldsymbol{X} = \left[\begin{array}{cc|c} 0.6 & -0.32 & -0.1 \\ 0.19 & 0.25 & 0.54 \\ \hline 0.1 & 0.1 & 0.16 \end{array}\right], \qquad (20a)$$

$$\boldsymbol{X}_d = \left[\begin{array}{cc|c} 0.1 & 0.01 & 0.03 \\ 0.11 & 0.05 & -0.12 \\ \hline 0.02 & 0.06 & 0.15 \end{array}\right], \qquad (20b)$$

$$\boldsymbol{P}_0 = \boldsymbol{P}_1 = \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix}, \ \boldsymbol{Q}_0 = \begin{bmatrix} 0.01 & 0 & 0, \end{bmatrix}, \ \boldsymbol{Q}_1 = \begin{bmatrix} 0 & 0.01 & 0 \end{bmatrix}, \qquad (20c)$$

$$d_1^h = 3, \ d_1^v = 2, \ d_2^v = 6. \qquad (20d)$$

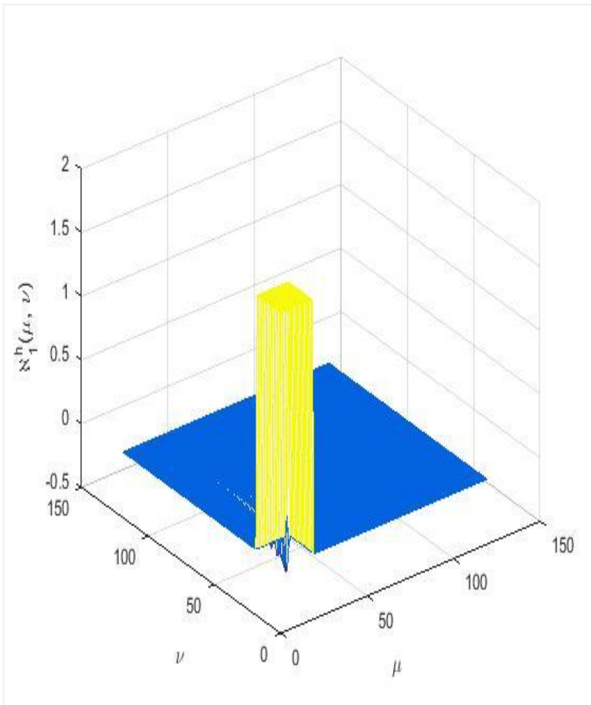
This example was considered in [17] for the GAS analysis of uncertain 2-*D* DSs employing various combinations of quantization and overflow. Now, for this example, our aim is to obtain different set of values of d_2^h for a given distinct set of values of d_1^h by iteratively solving the conditions presented in Theorem 1 w.r.t. d_2^h. With the MATLAB software [37] along with YALMIP 3.0 [52], it is checked that Theorem 1 yields solutions for $d_2^h = 18$. Thus, the GAS of the considered system is assured over $3 \le d^h(\mu) \le 18$ and $2 \le d^v(\nu) \le 6$.

With $\mathrm{T}_0 = \mathrm{T}_1 = 1$, $d^h(\mu) = 3 + \lfloor |15\sin(180(\mu-1)/\pi|\rceil$ and $d^v(\nu) = 2 + \lfloor |4\sin(180(\nu-1)/\pi|\rceil$, the trajectories of three state variables of the system given by Equation 1 are shown in *Figure 2*. The plots of TVDs $d^h(\mu)$ and $d^v(\nu)$ are shown in *Figure 3*. For the state trajectories, we have selected the initial conditions as given in Equation 21a and Equation 21b.

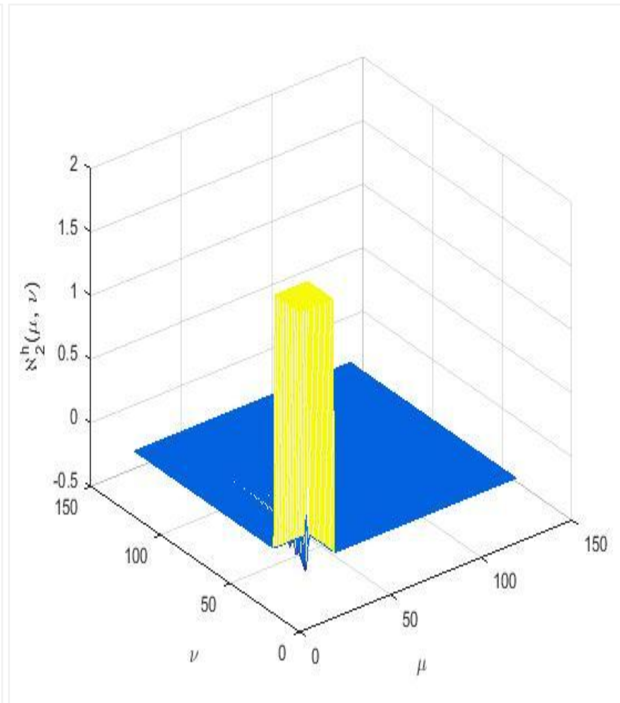
$$\aleph^h(\mu, \ \nu) = \begin{bmatrix} \aleph_1^h(\mu, \ \nu) & \aleph_2^h(\mu, \ \nu) \end{bmatrix}^{\mathrm{T}} = \begin{cases} \begin{bmatrix} 2 & 4 \end{bmatrix}^{\mathrm{T}}, & \forall \ 0 \le \nu < 19, \ 0 \le \mu \le 18 \\ \mathbf{0}, & \forall \ \nu \ge 19, \ 0 \le \mu \le 18, \end{cases} \qquad (21a)$$

$$\aleph_1^v(\mu, \ \nu) = \begin{cases} -3, & \forall \ 0 \le \mu < 19, \ 0 \le \nu \le 6 \\ 0, & \forall \ \mu \ge 19, \ 0 \le \nu \le 6. \end{cases} \qquad (21b)$$

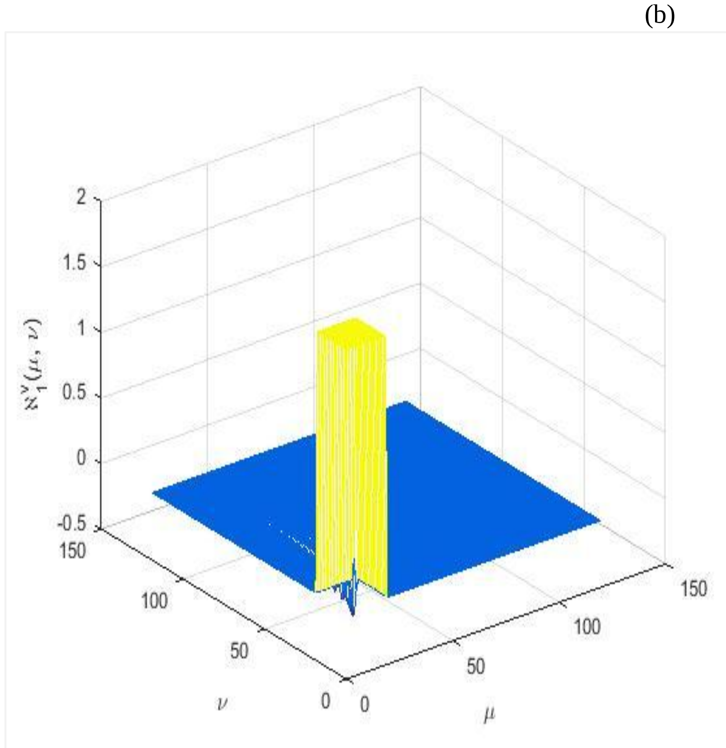
Figure 2 shows that the system trajectories tend toward the origin as $\mu + \nu \to \infty$, which is in accordance with the GAS of the system under study.



(a)

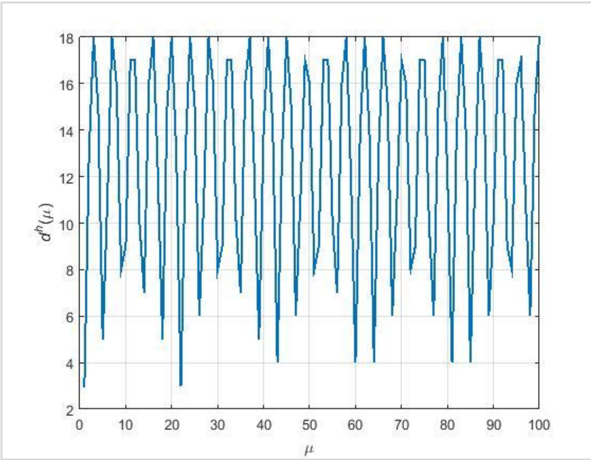


(b)

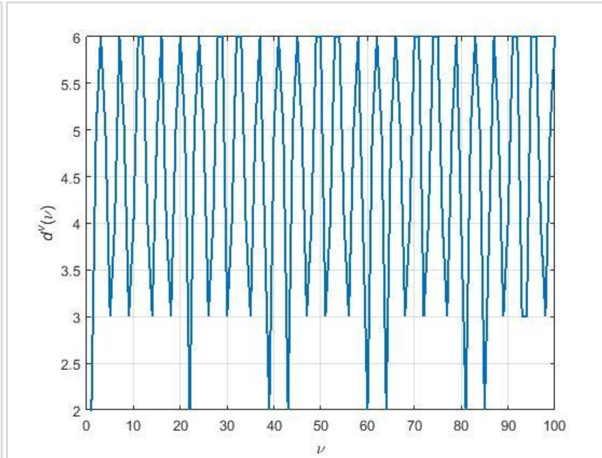


(c)

Figure 2 State trajectories of example 1



(a)



(b)

Figure 3 TVDs employed in the simulation for example 1

Example 2: Consider the 2-*D* DS described by Equation 17 together with Equations 22a-22c.

$$\boldsymbol{X} = \left[\begin{array}{cc|c} 0.7 & -0.2 & -0.12 \\ 0.23 & 0.3 & 0.4 \\ \hline 0.14 & 0.1 & 0.17 \end{array}\right], \quad (22a)$$

$$\boldsymbol{X}_d = \left[\begin{array}{cc|c} 0.09 & 0.01 & 0.02 \\ 0.13 & 0.06 & -0.1 \\ \hline 0.02 & 0.06 & 0.18 \end{array}\right], \quad (22b)$$

$$d_1^v = 2, \; d_2^v = 6. \quad (22c)$$

For some given values of d_1^h, the different values of d_2^h obtained (which affirm the GAS of the considered system) are presented in *Table 1*. *Table 1* shows that, when compared to [9], Corollary 1 produces a more relaxed result for the present example. The trajectories of three states of the system given by Equation 17 are shown in *Figure 4*. For the state trajectories, we have taken the initial conditions as shown in Equation 23a and Equation 23b.

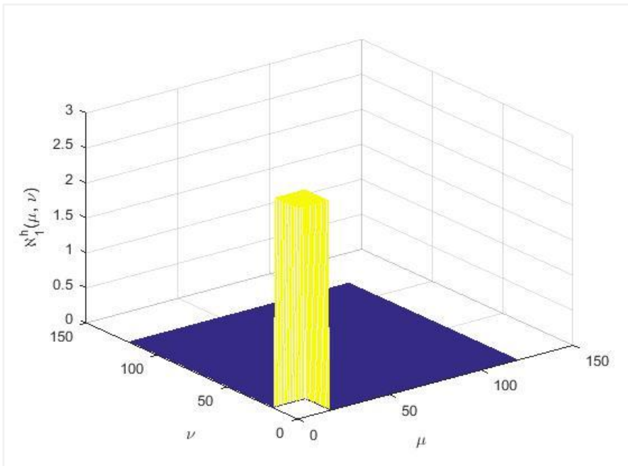
Table 1 Upper delay bound d_2^h for various d_1^h in Example 2

Methods/ d_1^h for $2 \le d^v(\nu) \le 6$	3	5	7	9
Theorem 4 [9]	13	15	17	19
Corollary 1 (Proposed)	16	18	20	22

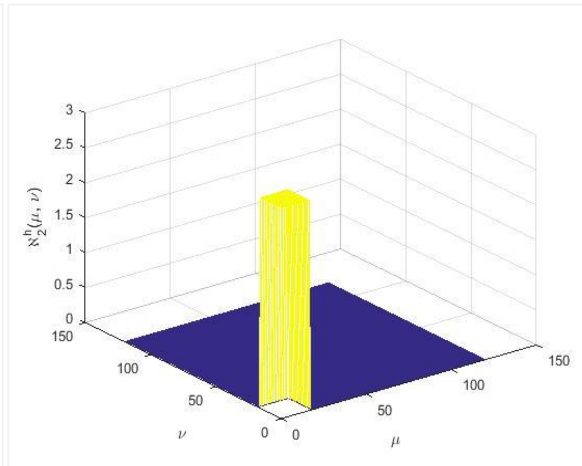
$$\aleph^h(\mu,\ \nu) = \left[\aleph_1^h(\mu,\ \nu) \quad \aleph_2^h(\mu,\ \nu)\right]^{\mathrm{T}} = \begin{cases} \begin{bmatrix} 3 & 5 \end{bmatrix}^{\mathrm{T}}, & \forall\ 0 \le \nu < 17,\ 0 \le \mu \le 16 \\ \mathbf{0}, & \forall\ \nu \ge 17,\ 0 \le \mu \le 16, \end{cases} \quad (23a)$$

$$\aleph_1^v(\mu,\ \nu) = \begin{cases} -1, & \forall\ 0 \le \mu < 17,\ 0 \le \nu \le 6 \\ 0, & \forall\ \mu \ge 17,\ 0 \le \nu \le 6. \end{cases} \quad (23b)$$

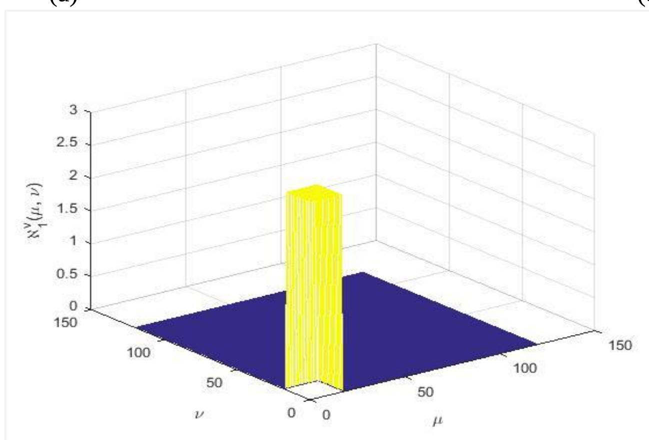
It is clear from *Figure 4* that all the system trajectories converge to origin as $\mu + \nu \to \infty$. It supports the GAS for this example.



(a)



(b)



(c)

Figure 4 State trajectories of example 2

5.Discussion

Although the underlying system in Equations 1-5 considers norm-bounded parameter uncertainties, further research is needed to extend the presented approach to 2-*D* delayed systems with polytopic uncertainties and SNL.

The matrix variables $\boldsymbol{Y}^h$ and $\boldsymbol{Y}^v$ play a key role in reducing the conservatism of the presented criterion. The special selection of these matrix variables as the diagonal ones, helps in reducing the computational complexity.

The validity of the conditions in Theorem 1 and Corollary 1 can be easily established using MATLAB software [37] and YALMIP 3.0 [52].

Many existing approaches (see, for example, [32, 33, 37−40]) have ignored the effects of SNL while studying the stability behaviour of 2-*D* delayed systems. On the other hand, the presented criterion (Theorem 1) can be applied to verify the GAS of 2-*D* delayed DSs in the Roesser model with SNL and TVDs. To estimate the sum terms in the forward difference of the LKF, the WBI technique is combined with RCI. Further, a new stability criterion (Corollary 1) is proposed, which yields more relaxed outcomes than the previous criterion [9].

The proposed criteria provide only sufficient conditions for GAS of the 2-*D* system. Further investigation is required to lessen the conservatism of the results obtained in this paper.

A complete list of abbreviations is shown in *Appendix II.*

6.Conclusion and future work

By employing the idea of RCI along with WBI, a new criterion for the GAS of uncertain 2-*D* DSs based on the Roesser model with TVDs and SNL has been presented. In addition, a stability condition for the 2-*D* delayed DS without uncertainties is brought out which provides improved results as compared to [9].

The possibility of extending the presented approach to address stability problems of H_∞ robust filtering for 2-*D* DSs [15], 2-*D* systems with external interference [54] and delay, etc., appears to be an appealing problem for future investigation.

Acknowledgment

None.

Conflicts of interest

The authors have no conflicts of interest to declare.

Author's contribution statement

Dinesh Chaurasia: Conceptualization, formal analysis, methodology, validation and writing draft. **Kalpana Singh:** Conceptualization, formal analysis, methodology, validation and writing draft. **V. Krishna Rao Kandanvli:** Conceptualization, formal analysis, methodology and supervision. **Haranath Kar:** Conceptualization, formal analysis, methodology and supervision.

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Appendix I

Proof of Corollary 1 With $\Delta \boldsymbol{X} = \Delta \boldsymbol{X}_d = \boldsymbol{0}$ (i.e., no uncertainty) in Equations 1-5 and following the identical steps as given in the proof of Theorem 1, one can easily obtain Corollary 1.

Appendix II

S. No.	Abbreviation	Description
1	2-D	Two-Dimensional
2	DS	Discrete System
3	FMSLSS	Fornasini-Marchesini Second Local State Space
4	FWM	Free Weighting Matrix
5	GAS	Global Asymptotic Stability
6	HD	Horizontal Direction
7	JI	Jensen Inequality
8	LKF	Lyapunov-Krasovskii Function
9	LMI	Linear Matrix Inequality
10	RCI	Reciprocal Convex Inequality
11	SNL	Saturation Nonlinearities
12	TVD	Time-Varying Delay
13	VD	Vertical Direction
14	WBI	Wirtinger-based Inequality

Appendix III

Proof of Theorem 1 Suppose the terms $\boldsymbol{\kappa}^h(\mu,\ \nu)$, $\boldsymbol{\kappa}^v(\mu,\ \nu)$ and $\varsigma(\mu,\ \nu)$ are given by Equation 24a, Equation 24b and Equation 25, respectively.

$$\boldsymbol{\kappa}^h(\mu,\ \nu)=\aleph^h(\mu+1,\ \nu)-\aleph^h(\mu,\ \nu)=\boldsymbol{f}^h(\boldsymbol{\sigma}^h(\mu,\ \nu))-\aleph^h(\mu,\ \nu), \tag{24a}$$

$$\boldsymbol{\kappa}^v(\mu,\ \nu)=\aleph^v(\mu,\ \nu+1)-\aleph^v(\mu,\ \nu)=\boldsymbol{f}^v(\boldsymbol{\sigma}^v(\mu,\ \nu))-\aleph^v(\mu,\ \nu), \tag{24b}$$

$$\varsigma(\mu,\ \nu)=col\{\aleph(\mu,\ \nu),\ \aleph_1(\mu,\ \nu),\ \aleph_2(\mu,\ \nu),\ \aleph_3(\mu,\ \nu),\ \aleph_4(\mu,\ \nu),\ \aleph_5(\mu,\ \nu),\ \aleph_6(\mu,\ \nu),\ \boldsymbol{f}(\boldsymbol{\sigma}(\mu,\ \nu))\}, \tag{25}$$

where

$$\aleph(\mu,\ \nu)=\begin{bmatrix}\aleph^h(\mu,\ \nu)\\ \hdashline \aleph^v(\mu,\ \nu)\end{bmatrix},\ \aleph_1(\mu,\ \nu)=\begin{bmatrix}\aleph^h(\mu-d^h(\mu),\ \nu)\\ \hdashline \aleph^v(\mu,\ \nu-d^v(\nu))\end{bmatrix},\quad \aleph_2(\mu,\ \nu)=\begin{bmatrix}\aleph^h(\mu-d_1^h,\ \nu)\\ \hdashline \aleph^v(\mu,\ \nu-d_1^v)\end{bmatrix},\ \aleph_3(\mu,\ \nu)=\begin{bmatrix}\aleph^h(\mu-d_2^h,\ \nu)\\ \hdashline \aleph^v(\mu,\ \nu-d_2^v)\end{bmatrix},$$

$$\aleph_4(\mu,\ \nu)=\begin{bmatrix}\chi(\mu,\ 0,\ d_1^h)\\ \hdashline \chi(\nu,\ 0,\ d_1^v)\end{bmatrix},\ \aleph_5(\mu,\ \nu)=\begin{bmatrix}\chi(\mu,\ d_1^h,\ d^h(\mu))\\ \hdashline \chi(\nu,\ d_1^v,\ d^v(\nu))\end{bmatrix},\quad \aleph_6(\mu,\ \nu)=\begin{bmatrix}\chi(\mu,\ d^h(\mu),\ d_2^h)\\ \hdashline \chi(\nu,\ d^v(\nu),\ d_2^v)\end{bmatrix}.$$

A 2-*D* quadratic LKF taken into consideration and is given by Equations 26-28:

$$V(\aleph(\mu,\ \nu))=V(\aleph^h(\mu,\ \nu))+V(\aleph^v(\mu,\ \nu))=V_1(\aleph(\mu,\ \nu))+V_2(\aleph(\mu,\ \nu))+V_3(\aleph(\mu,\ \nu)) \tag{26}$$

with

$$V_1(\aleph(\mu,\ \nu))=\xi^{h^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{W}^h\xi^h(\mu,\ \nu)+\xi^{v^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{W}^v\xi^v(\mu,\ \nu), \tag{27a}$$

$$\begin{aligned}V_2(\aleph(\mu,\ \nu))=&\sum_{r=\mu-d_1^h}^{\mu-1}\aleph^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{H}_1^h\aleph^h(r,\ \nu)+\sum_{r=\mu-d_2^h}^{\mu-1}\aleph^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{H}_2^h\aleph^h(r,\ \nu)\\ &+\sum_{s=-d_2^h}^{-d_1^h}\sum_{r=\mu+s}^{\mu-1}\aleph^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{H}_3^h\aleph^h(r,\ \nu)+\sum_{r=\nu-d_1^v}^{\nu-1}\aleph^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{H}_1^v\aleph^v(\mu,\ r)\\ &+\sum_{r=\nu-d_2^v}^{\nu-1}\aleph^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{H}_2^v\aleph^v(\mu,\ r)+\sum_{s=-d_2^v}^{-d_1^v}\sum_{r=\nu+s}^{\nu-1}\aleph^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{H}_3^v\aleph^v(\mu,\ r),\end{aligned} \tag{27b}$$

$$\begin{aligned}V_3(\aleph(\mu,\ \nu))=&\,d_1^h\sum_{s=-d_1^h+1}^{0}\sum_{r=\mu-1+s}^{\mu-1}\boldsymbol{\kappa}^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{T}_1^h\boldsymbol{\kappa}^h(r,\ \nu)+d_{12}^h\sum_{s=-d_2^h}^{-d_1^h}\sum_{r=\mu-1+s}^{\mu-1}\boldsymbol{\kappa}^{h^{\mathrm{T}}}(r,\ \nu)\boldsymbol{T}_2^h\boldsymbol{\kappa}^h(r,\ \nu)\\ &+d_1^v\sum_{s=-d_1^v+1}^{0}\sum_{r=\nu-1+s}^{\nu-1}\boldsymbol{\kappa}^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{T}_1^v\boldsymbol{\kappa}^v(\mu,\ r)+d_{12}^v\sum_{s=-d_2^v}^{-d_1^v}\sum_{r=\nu-1+s}^{\nu-1}\boldsymbol{\kappa}^{v^{\mathrm{T}}}(\mu,\ r)\boldsymbol{T}_2^v\boldsymbol{\kappa}^v(\mu,\ r),\end{aligned} \tag{27c}$$

where

$$\xi^h(\mu,\ \nu)=\begin{bmatrix}\aleph^{h^{\mathrm{T}}}(\mu,\ \nu) & \sum_{r=\mu-d_1^h}^{\mu-1}\aleph^{h^{\mathrm{T}}}(r,\ \nu) & \sum_{r=\mu-d_2^h}^{\mu-d_1^h-1}\aleph^{h^{\mathrm{T}}}(r,\ \nu)\end{bmatrix}^{\mathrm{T}}, \tag{28a}$$

$$\xi^v(\mu,\ \nu)=\begin{bmatrix}\aleph^{v^{\mathrm{T}}}(\mu,\ \nu) & \sum_{r=\nu-d_1^v}^{\nu-1}\aleph^{v^{\mathrm{T}}}(\mu,\ r) & \sum_{r=\nu-d_2^v}^{\nu-d_1^v-1}\aleph^{v^{\mathrm{T}}}(\mu,\ r)\end{bmatrix}^{\mathrm{T}}. \tag{28b}$$

The above 2-*D* LKF can be considered as an extension of 1-*D* LKF employed in [26]. The forward difference of the LKF shown in Equation 26 along the trajectories of the system (Equation 1) is given by Equation 29 and Equation 30:

$$\begin{aligned}\Delta V(\aleph(\mu,\ \nu))&=V(\aleph^h(\mu+1,\ \nu))-V(\aleph^h(\mu,\ \nu))+V(\aleph^v(\mu,\ \nu+1))-V(\aleph^v(\mu,\ \nu))\\ &=\Delta V_1(\aleph(\mu,\ \nu))+\Delta V_2(\aleph(\mu,\ \nu))+\Delta V_3(\aleph(\mu,\ \nu)),\end{aligned} \tag{29}$$

where

$$\begin{aligned}\Delta V_1(\aleph(\mu,\ \nu))&=\xi^{h^{\mathrm{T}}}(\mu+1,\ \nu)\boldsymbol{W}^h\xi^h(\mu+1,\ \nu)-\xi^{h^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{W}^h\xi^h(\mu,\ \nu)+\xi^{v^{\mathrm{T}}}(\mu,\ \nu+1)\boldsymbol{W}^v\xi^v(\mu,\ \nu+1)-\xi^{v^{\mathrm{T}}}(\mu,\ \nu)\boldsymbol{W}^v\xi^v(\mu,\ \nu)\\ &=\varsigma^{\mathrm{T}}(\mu,\ \nu)\tilde{\varnothing}(d^h(\mu),\ d^v(\nu))\varsigma(\mu,\ \nu),\end{aligned} \tag{30a}$$

$$\tilde{\varnothing}(d^h(\mu),\ d^v(v)) =$$

$$\begin{bmatrix} -W_1+(W_2+W_2^{\mathrm{T}})/2 & \mathbf{0} & (W_3-W_2)/2 & -W_3/2 & \varnothing_{15} & \varnothing_{16} & \varnothing_{17} & W_2^{\mathrm{T}}/2 \\ * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ * & * & \mathbf{0} & \mathbf{0} & \varnothing_{35} & \varnothing_{36} & \Upsilon_3(W_6-W_5)/2 & (W_3^{\mathrm{T}}-W_2^{\mathrm{T}})/2 \\ * & * & * & \mathbf{0} & -\Upsilon_1(W_5^{\mathrm{T}}/2) & -\Upsilon_2(W_6/2) & -\Upsilon_3(W_6/2) & -W_3^{\mathrm{T}}/2 \\ * & * & * & * & \mathbf{0} & \mathbf{0} & \mathbf{0} & \Upsilon_1(W_2^{\mathrm{T}}/2) \\ * & * & * & * & * & \mathbf{0} & \mathbf{0} & \Upsilon_2(W_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & \mathbf{0} & \Upsilon_3(W_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & * & W_1 \end{bmatrix}, \quad (30b)$$

$$\begin{aligned} \Delta V_2(\aleph(\mu,\ v)) = {} & \aleph^{h^{\mathrm{T}}}(\mu,\ v)H_1^h\aleph^h(\mu,\ v)+\aleph^{v^{\mathrm{T}}}(\mu,\ v)H_1^v\aleph^v(\mu,\ v) \\ & +\aleph^{h^{\mathrm{T}}}(\mu,\ v)H_2^h\aleph^h(\mu,\ v)+\aleph^{v^{\mathrm{T}}}(\mu,\ v)H_2^v\aleph^v(\mu,\ v) \\ & +\aleph^{h^{\mathrm{T}}}(\mu,\ v)H_3^h\aleph^h(\mu,\ v)+d_{12}^h\aleph^{h^{\mathrm{T}}}(\mu,\ v)H_3^h\aleph^h(\mu,\ v) \\ & +\aleph^{v^{\mathrm{T}}}(\mu,\ v)H_3^v\aleph^v(\mu,\ v)+d_{12}^v\aleph^{v^{\mathrm{T}}}(\mu,\ v)H_3^v\aleph^v(\mu,\ v) \\ & -\aleph^{h^{\mathrm{T}}}(\mu-d_1^h,\ v)H_1^h\aleph^h(\mu-d_1^h,\ v)-\aleph^{v^{\mathrm{T}}}(\mu,\ v-d_1^v)H_1^v\aleph^v(\mu,\ v-d_1^v) \\ & -\aleph^{h^{\mathrm{T}}}(\mu-d_2^h,\ v)H_2^h\aleph^h(\mu-d_2^h,\ v)-\aleph^{v^{T}}(\mu,\ v-d_2^v)H_2^v\aleph^v(\mu,\ v-d_2^v)\,, \end{aligned} \quad (30c)$$

$$\begin{aligned} \Delta V_3(\aleph(\mu,\ v)) = {} & \kappa^{h^{\mathrm{T}}}(\mu,\ v)(d_1^{h^2}T_1^h)\kappa^h(\mu,\ v)+\kappa^{h^{\mathrm{T}}}(\mu,\ v)(d_{12}^{h^2}T_2^h)\kappa^h(\mu,\ v)+\kappa^{v^{\mathrm{T}}}(\mu,\ v)(d_1^{v^2}T_1^v)\kappa^v(\mu,\ v) \\ & +\kappa^{v^{\mathrm{T}}}(\mu,\ v)(d_{12}^{v^2}T_2^v)\kappa^v(\mu,\ v)+\sum_{i=1}^{2}(S_i^h(\mu,\ v)+S_i^v(\mu,\ v)), \end{aligned} \quad (30d)$$

$$S_1^h(\mu,\ v) = -d_1^h\sum_{s=\mu-d_1^h}^{\mu-1}\kappa^{h^{\mathrm{T}}}(s,\ v)T_1^h\kappa^h(s,\ v), \quad (30e)$$

$$S_1^v(\mu,\ v) = -d_1^v\sum_{s=v-d_1^v}^{v-1}\kappa^{v^{\mathrm{T}}}(\mu,\ s)T_1^v\kappa^v(\mu,\ s), \quad (30f)$$

$$S_2^h(\mu,\ v) = -d_{12}^h\sum_{s=\mu-d_2^h}^{\mu-d_1^h-1}\kappa^{h^{\mathrm{T}}}(s,\ v)T_2^h\kappa^h(s,\ v), \quad (30g)$$

$$S_2^v(\mu,\ v) = -d_{12}^v\sum_{s=v-d_2^v}^{v-d_1^v-1}\kappa^{v^{\mathrm{T}}}(\mu,\ s)T_2^v\kappa^v(\mu,\ s). \quad (30h)$$

By employing Lemmas 1, 2 and following [26], one can show that the inequality given by Equations 31-34 holds:

$$\Delta V(\aleph(\mu,v)) \leqslant \varsigma^T(\mu,v)\Xi(d^h(\mu),d^v(v))\varsigma(\mu,v) - \beta \quad (31)$$

where

$$\beta = \sigma^{\mathrm{T}}(\mu,\ v)Cf(\sigma(\mu,\ v)) + f^{\mathrm{T}}(\sigma(\mu,\ v))C^{\mathrm{T}}\sigma(\mu,\ v) - f^{\mathrm{T}}(\sigma(\mu,\ v))(C+C^{\mathrm{T}})f(\sigma(\mu,\ v)), \quad (32)$$

$$\Xi(d^h(\mu),\ d^v(v)) =$$

$$\begin{bmatrix} \varnothing_{11} & \mathbf{0} & ((\boldsymbol{W}_3-\boldsymbol{W}_2)/2)-2\boldsymbol{T}_1 & -\boldsymbol{W}_3/2 & \varnothing_{15}+3\boldsymbol{T}_1 & \varnothing_{16} & \varnothing_{17} & \varnothing_{18}+(\boldsymbol{W}_2^{\mathrm{T}}/2)+\bar{\boldsymbol{X}}^{\mathrm{T}}\boldsymbol{C} \\ * & \varnothing_{22} & \varnothing_{23} & \varnothing_{24} & \mathbf{0} & \varnothing_{26} & \varnothing_{27} & \bar{\boldsymbol{X}}_d^{\mathrm{T}}\boldsymbol{C} \\ * & * & -\boldsymbol{H}_1-4(\boldsymbol{T}_1+\boldsymbol{T}_2) & \varnothing_{34} & \varnothing_{35}+3\boldsymbol{T}_1 & \varnothing_{36}+3\boldsymbol{T}_2 & \varnothing_{37} & (\boldsymbol{W}_3^{\mathrm{T}}-\boldsymbol{W}_2^{\mathrm{T}})/2 \\ * & * & * & -\boldsymbol{H}_2-4\boldsymbol{T}_2 & -\Upsilon_1(\boldsymbol{W}_5^{\mathrm{T}}/2) & \varnothing_{46} & \varnothing_{47} & -\boldsymbol{W}_3^{\mathrm{T}}/2 \\ * & * & * & * & -3\boldsymbol{T}_1 & \mathbf{0} & \mathbf{0} & \Upsilon_1(\boldsymbol{W}_2^{\mathrm{T}}/2) \\ * & * & * & * & * & -3\boldsymbol{T}_2 & -\boldsymbol{Y}_{22} & \Upsilon_2(\boldsymbol{W}_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & -3\boldsymbol{T}_2 & \Upsilon_3(\boldsymbol{W}_3^{\mathrm{T}}/2) \\ * & * & * & * & * & * & * & \boldsymbol{W}_1-\varnothing_{18}-(\boldsymbol{C}+\boldsymbol{C}^{\mathrm{T}}) \end{bmatrix}, \quad (33)$$

$$\bar{\boldsymbol{X}} = \boldsymbol{X}+\Delta\boldsymbol{X}, \quad \bar{\boldsymbol{X}}_d = \boldsymbol{X}_d+\Delta\boldsymbol{X}_d. \quad (34)$$

In the light of Equation 3, the term β (see Equation 32) is non-negative [10]. Thus, the condition $\Xi(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ along with Equation 10 implies $\Delta V(\aleph(\mu,\ \nu)) < 0$ which assures the GAS of the system given by Equations 1-5.

Similar to [10], it is apparent that $\lim\limits_{\mu\to\infty \text{ and/or } \nu\to\infty} \aleph(\mu,\ \nu) = \lim\limits_{\mu+\nu\to\infty} \aleph(\mu,\ \nu) = \mathbf{0}$ because of boundary conditions represented in Equation 5.

Next, in view of Equation 4 and Lemma 3, it can easily be shown that the condition $\Xi(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ is mathematically equivalent to $\varnothing(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$. Observe that, $\varnothing(d^h(\mu),\ d^v(\nu))$ is an affine function with respect to $d^h(\mu)$ and $d^v(\nu)$. Therefore, the condition $\varnothing(d^h(\mu),\ d^v(\nu)) < \mathbf{0}$ is satisfied iff Equation 11 holds true. This completes the proof.