Stability of \( n \)– Homomorphisms, \( n \)– Derivations of A \( n \)– Dimensional Additive Functional Equation in \( C^* \)– Ternary Algebras

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Abstract

In this paper, the authors established the generalized Ulam-Hyers stability of \( n \)– Homomorphisms, \( n \)– Derivations of a \( n \)– dimensional additive functional equation

\[
\sum_{i=1}^{n} g \left( \sum_{j=1}^{i} x_j \right) = \sum_{i=1}^{n} (n-i+1) g \left( x_i \right),
\]

where \( n \geq 2 \) on \( C^* \)–ternary algebras.

Keywords

Additive functional equation, homomorphism, derivations, \( C^* \)ternary algebra.

1. Introduction and Preliminaries

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [2]. It was further generalized and excellent results obtained by number of authors [3,4,5,6,7,8]. During the past two decades, a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings, for example, Cauchy-Jensen mappings, k-additive mappings, invariant means, multiplicative mappings, bounded nth differences, convex functions, generalized orthogonality mappings, Euler-Lagrange functional equations, differential equations, and Navier-Stokes equations (see [9 -15]).

Also, the stability problem of ternary homomorphisms and ternary derivations was established by Park [16] and J.M.Rassias, Kim [17]

Definition 1.1 [18] A \( C^* \)– ternary algebra is a complex Banach space \( A \), equipped with a ternary product \( (x, y, z) \rightarrow [x, y, z] \) of \( A^3 \) into \( A \), which is \( C \)–linear in the outer variables, conjugate \( C \)–linear in the middle variable, and associative in the sense that

\[
[x, y, [z, w, v]] = [x, [w, z, y], v] = [[x, y, z], w, v],
\]

and satisfies

\[
\| [x, y, z] \| = \| x \| \| y \| \| z \| \quad \text{and} \quad \| [x, x, x] \| = m \| x \|^3.
\]

Every left Hilbert \( C^* \)–module is a \( C^* \)–ternary algebra via the ternary product \( [x, y, z] = (x, y) z \). If a \( C^* \)–ternary algebra \( (A, \{\cdot, \cdot, \cdot\}) \) has an identity, i.e., an element \( e \in A \) such that \( x = [x, e, e] = [e, e, x] \) for all \( x \in A \), then it is routine to verify that \( A \), endowed with \( x \circ y = [x, e, y] \) and \( x^* = [e, e, x] \), is a unital \( C^* \)–algebra. Conversely, if \( (A, \circ) \) is a unital \( C^* \)–algebra, then \( [x, y, z] = x \circ y \circ z \) makes \( A \) into a \( C^* \)–ternary algebra.

Definition 1.2 [19, 20] Let \( A \) and \( B \) be \( C^* \)–ternary algebras. A \( C \)–linear mapping \( H : A \rightarrow B \) is called a \( C^* \)–ternary algebra homomorphism if

\[
H([x, y, z]) = [H(x), H(y), H(z)]
\]

for all \( x, y, z \in A \). If, in addition, the mapping \( H \) is bijective, then the mapping \( H : A \rightarrow B \) is called a \( C^* \)–ternary algebra isomorphism.

Definition 1.3 [19, 20] A \( C \)–linear mapping \( H : A \rightarrow A \) is called a \( C^* \)– ternary derivation if

\[
\delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)].
\]

Definition 1.4 \( C^* \)– ternary algebra \( n \)– homomorphism Let \( A \) and \( B \) be \( C^* \)– ternary
algebras. A $C$–linear mapping $H : A \to B$ is called a $C^*$–ternary algebra homomorphism if $H([x_1, \ldots, x_n]) = [H(x_1), H(\ldots), H(x_n)]$ for all $x_1, \ldots, x_n \in A$.

**Definition 1.5** $C^*$–ternary algebra $n$–derivation

A $C$–linear mapping $H : A \to A$ is called a $C^*$–ternary derivation if $\delta([x_1, \ldots, x_n]) = [\delta(x_1), \ldots, x_n] + \ldots + [x_1, \ldots, \delta(x_n)]$.

In this paper, the authors proved the generalized Ulam-Hyers stability of a $n$–dimensional additive functional equation

$$\sum_{i=1}^{n} g \left( \sum_{j=1}^{i} x_j \right) = \sum_{i=1}^{n} (n-i+1) g(x_i), \quad (1.1)$$

where $n \geq 2$ on Banach algebras.

In Section 2 and Section 3, the generalized Ulam-Hyers stability of $n$–homomorphisms and $n$–derivations of a $n$–dimensional additive functional equation (1.1), is respectively provided.

Through out this paper, let us consider $X$ and $Y$ to be a $C^*$–ternary algebra with norm $\| \cdot \|_X$ and a $C^*$–ternary algebra with norm $\| \cdot \|_Y$ respectively.

**2. $n$–Homomorphisms Stability Results**

In this section, the generalized Ulam-Hyers stability of $n$–homomorphisms of the additive functional equation (1.1) is provided.

**Theorem 2.1** Let $j \in \{-1, 1\}$. Assume $\alpha : X^n \to [0, \infty)$ and $\beta : X^n \to [0, \infty)$ be functions such that

$$\lim_{n \to \infty} \frac{\alpha(2^nx_1, \ldots, 2^nx_n)}{2^n} = 0, \quad (2.1)$$

$$\lim_{n \to \infty} \frac{\beta(2^nx_1, \ldots, 2^nx_n)}{2^n} = 0 \quad (2.2)$$

for all $x_1, \ldots, x_n \in X$. Let $g : X \to Y$ be a function satisfying the inequality

$$\left\| \sum_{i=1}^{n} g \left( \sum_{j=1}^{i} x_j \right) - \sum_{i=1}^{n} (n-i+1) g(x_i) \right\|_Y \leq \alpha(x_1, \ldots, x_n) \quad (2.3)$$

$$\left\| g([x_1, \ldots, x_n]) - [g(x_1), \ldots, g(x_n)] \right\|_Y \leq \beta(x_1, \ldots, x_n) \quad (2.4)$$

for all $x_1, \ldots, x_n \in X$. Then there exists a unique $C^*$–ternary algebra $n$–homomorphism mapping $H : X \to Y$ such that

$$\left\| g(x) - H(x) \right\|_Y \leq \frac{1}{2} \sum_{k=1}^{\infty} \frac{\alpha(2^k x, 2^k x, 0, \ldots, 0)}{2^n} \quad (2.5)$$

for all $x \in X$.

**Proof.** Assume $j = 1$. Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (2.3), we get

$$\left\| g(x) - \frac{g(2x)}{2} \right\|_Y \leq \frac{1}{2} \alpha(x, x, 0, \ldots, 0) \quad (2.6)$$

for all $x \in X$. Now replacing $x$ by $2x$ and dividing by 2 in (2.6), we get

$$\left\| \frac{g(2x)}{2} - \frac{g(2^2 x)}{2^2} \right\|_Y \leq \frac{1}{2^2(n-1)} \alpha(2x, 2x, 0, \ldots, 0) \quad (2.7)$$

for all $x \in X$. From (2.6) and (2.7), we obtain

$$\left\| g(x) - \frac{g(2^j x)}{2^j} \right\|_Y \leq \left[ \left\| g(x) - \frac{g(2x)}{2} \right\|_Y + \left\| \frac{g(2x)}{2} - \frac{g(2^2 x)}{2^2} \right\|_Y \right] \leq \frac{1}{2(n-1)} \alpha(x, x, 0, \ldots, 0) \quad (2.8)$$

for all $x \in X$.

In general for any positive integer $k$, we get

$$\left\| g(x) - \frac{g(2^k x)}{2^k} \right\|_Y.$$
\[
\left\| \frac{g(2^i x) - g(2^{i+\ell} x)}{2^i} \right\|_\infty \leq \frac{1}{2(n-1)} \sum_{i=0}^{k-1} \alpha(2^i x, 2^i x, 0, \ldots, 0) 2^i
\]

for all \( x \in X \). In order to prove the convergence of the sequence \( \left\{ g(2^k x) \right\} \), replace \( x \) by \( 2^\ell x \) and dividing by \( 2^\ell \) in (2.9), for any \( k, \ell > 0 \), we deduce

\[
\left\| \frac{g(2^\ell x) - g(2^{k+\ell} x)}{2^{i+\ell}} \right\|_\infty \leq \frac{1}{2(n-1)} \sum_{i=0}^{k-1} \alpha(2^{i+\ell} x, 2^{i+\ell} x, 0, \ldots, 0) 2^{i+\ell}
\]

\[
\leq \frac{1}{2(n-1)} \sum_{i=0}^{\infty} \alpha(2^{i+\ell} x, 2^{i+\ell} x, 0, \ldots, 0) 2^{i+\ell}
\]

\[
\rightarrow 0 \text{ as } k \rightarrow \infty
\]

for all \( x \in X \). Hence the sequence \( \left\{ \frac{g(2^k x)}{2^k} \right\} \) is a Cauchy sequence. Since \( Y \) is complete, there exists a mapping \( H : X \rightarrow Y \) such that

\[
H(x) = \lim_{n \rightarrow \infty} \frac{g(2^k x)}{2^k}, \quad \forall \ x \in X.
\]

Letting \( k \rightarrow \infty \) in (2.9) we see that (2.5) holds for all \( x \in X \). To prove that \( H \) satisfies (1.1), replacing \( (x_1, \ldots, x_n) \) by \( (2^k x_1, \ldots, 2^k x_n) \) and dividing by \( 2^k \) in (2.3), we obtain

\[
\frac{1}{2^k} \left\| \sum_{j=1}^i g \left( \sum_{j=1}^i 2^k x_j \right) - \sum_{j=1}^n (n-i+1) g \left( 2^k x_j \right) \right\|_\infty
\]

\[
\leq \frac{1}{2} \alpha(2^k x_1, \ldots, 2^k x_n)
\]

for all \( x_1, \ldots, x_n \in X \).

Letting \( k \rightarrow \infty \) in the above inequality and using the definition of \( H(x) \), we see that

\[
\sum_{i=1}^n H \left( \sum_{j=1}^i x_j \right) = \sum_{i=1}^n (n-i+1) H(x_i).
\]

Hence \( H \) satisfies (1.1) for all \( x_1, \ldots, x_n \in X \). It follows from (2.4) that

\[
\|H([x_1, \ldots, x_n]) - [H(x_1), \ldots, H(x_n)]\|_\infty
\]

\[
\leq \frac{1}{2^k} \left\| \left\{ g(2^k x_1), \ldots, g(2^k x_n) \right\} \right\|_\infty
\]

\[
\rightarrow 0 \text{ as } k \rightarrow \infty
\]

for all \( x_1, \ldots, x_n \in X \). Hence \( H([x_1, \ldots, x_n]) = [H(x_1), \ldots, H(x_n)] \) for all \( x_1, \ldots, x_n \in X \). To prove that \( H \) is unique, let \( G(x) \) be another mapping satisfying (2.1) and (2.5), then

\[
\|H(x) - G(x)\|_\infty
\]

\[
\leq \frac{1}{2^k} \left\{ \left\| H(2^k x) - g(2^k x) \right\|_\infty + \left\| g(2^k x) - G(2^k x) \right\|_\infty \right\}
\]

\[
\leq \frac{1}{2^k} \left\{ \frac{1}{2} \sum_{i=0}^\infty \alpha(2^{i+k} x, 2^{i+k} x, 0, \ldots, 0) 2^{i+k} \right. \right. \]

\[
+ \left. \left. \frac{1}{2} \sum_{i=0}^\infty \alpha(2^{i+k} x, 2^{i+k} x, 0, \ldots, 0) 2^{i+k} \right\}
\]

\[
\leq \sum_{i=0}^\infty \alpha(2^{i+k} x, 2^{i+k} x, 0, \ldots, 0) 2^{i+k}
\]

\[
\rightarrow 0 \text{ as } k \rightarrow \infty
\]

for all \( x \in X \). Hence \( H \) is unique. Thus the mapping \( H : X \rightarrow Y \) is a unique \( C^* \) -ternary algebra \( H \) -homomorphism satisfying (2.5).

For \( j = -1 \), we can prove a similar stability result. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 2.1 concerning the stability of (1.1).

**Corollary 2.2** Let \( \lambda, \lambda_1 \) and \( \delta \) be nonnegative real numbers. Let a function \( g : X \rightarrow Y \) satisfies the inequality
The rest of the proof is similar tracing to that of Theorem 2.1.

\( \sum_{i=1}^{n} g \left( \sum_{j=1}^{i} x_j \right) - \sum_{i=1}^{n} (n-i+1) g \left( x_i \right) \)  

\[ \begin{array}{l}
\lambda, \\
\leq \lambda \sum_{i=1}^{n} \| x_i \|_X, \\
\leq \lambda \left\{ \prod_{i=1}^{n} \| x_i \|_X + \sum_{i=1}^{n} \| x_i \|_X^n \right\},
\end{array} \]

\( \| g((x_1, \ldots, x_n)) - [g(x_1), \ldots, g(x_n)] \|_Y \)

\( \| g(x) - H(x) \|_Y \leq \frac{\lambda}{(n-1) \left| 2 - 2^j \right|}, \quad \left(2.12\right) \)

for all \( x \in X \).

\[ \begin{array}{l}
\alpha \left( 2^{nj} x_1, \ldots, 2^{nj} x_n \right) = 0, \\
\beta \left( 2^{nj} x_1, \ldots, 2^{nj} x_n \right) = 0
\end{array} \]  

(3.1) (3.2)

for all \( x_1, \ldots, x_n \in X \). Let \( g : X \to Y \) be a function satisfying the inequality

\( \| g((x_1, x_2, \ldots, x_n)) - [g(x_1), x_2, \ldots, x_n] - [x_1, g(x_2), \ldots, x_n] \ldots - [x_1, x_2, \ldots, g(x_n)] \|_Y \leq \beta(x_1, x_2, \ldots, x_n) \)  

(3.3) (3.4)

for all \( x_1, \ldots, x_n \in X \). Then there exists a unique \( C^* \)-ternary algebra \( n \)-derivation mapping \( \delta : X \to Y \) such that

\( \| g(x) - \delta(x) \|_Y \leq \frac{1}{2} \sum_{k=2}^{n} \alpha(2^{kj} x, 2^{kj} x, 0, \ldots, 0) \)  

(3.5)

for all \( x \in X \).

Proof. It follows from (3.4) that

\( \| \delta((x_1, \ldots, x_n)) - [\delta(x_1), \ldots, x_n] \ldots - [x_1, \ldots, \delta(x_n)] \|_Y \)

\[ \leq \frac{1}{2^k} \| g([2^k x_1, \ldots, 2^k x_n]) - [\delta(2^k x_1), \ldots, 2^k x_n] \ldots - [2^k x_1, \ldots, \delta(2^k x_n)] \|_Y \]

\[ \leq \frac{1}{2^k} \beta(2^k x_1, \ldots, 2^k x_n) \]

\( \to 0 \quad \text{as} \quad k \to \infty \)

for all \( x_1, \ldots, x_n \in X \). Hence

\( \delta((x_1, \ldots, x_n)) = [\delta(x_1), \ldots, x_n] \ldots - [x_1, \ldots, \delta(x_n)] \)

for all \( x_1, \ldots, x_n \in X \). The rest of the proof is similar tracing to that of Theorem 2.1.
The following Corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.1).

**Corollary 3.2** Let $\lambda$, $\lambda_1$ and $s$ be nonnegative real numbers. Let a function $g : X \to Y$ satisfies the inequality
\[
\sum_{i=1}^{n} g \left( \sum_{j=1}^{i} x_j \right) - \sum_{i=1}^{n} (n-i+1) g(x_i) \leq \lambda, \quad s < 1 \quad \text{or} \quad s > 1
\]
\[
= \lambda \sum_{i=1}^{n} \| x_i \|_X, \quad s < 1 / 3 \quad \text{or} \quad s > 1 / 3;
\]
\[
\| g([x_1, \ldots, x_n]) - [g(x_1), \ldots, g(x_n)] - \cdots - [x_1, \ldots, g(x_n)] \|_Y
\]
\[
\leq \lambda_1, \quad s < 1 / 3 \quad \text{or} \quad s > 1 / 3
\]
\[
= \lambda_1 \sum_{i=1}^{n} \| x_i \|_X^s, \quad (3.7)
\]
\[
= \lambda_1 \left\{ \prod_{i=1}^{n} \| x_i \|_X^s + \sum_{i=1}^{n} \| x_i \|_X^{3s} \right\},
\]
for all $x_1, \ldots, x_n \in X$. Then there exists a unique $C^*$ ternary algebra $n$ derivation function $\delta : X \to Y$ such that
\[
\| g(x) - \delta(x) \|_Y \leq \begin{cases} 
\lambda, & n \geq 1, \\
\lambda \| x \|_X, & n = 1, \\
\lambda \| x \|_X^s, & n = 1, \\
\lambda \| x \|_X^{3s}, & n = 1
\end{cases}
\]
\[
\| g(x) - \delta(x) \|_Y \leq \begin{cases} 
\lambda, & n \geq 1, \\
\lambda \| x \|_X, & n = 1, \\
\lambda \| x \|_X^s, & n = 1, \\
\lambda \| x \|_X^{3s}, & n = 1
\end{cases}
\]
for all $x \in X$.

**4. Conclusion**

The additive function $g(x) = x$ is the solution of the additive functional equation (1.1), the functional equation can be rewritten as follows
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} (n-i+1)(x_i).
\]
That is
\[
x_1 + (x_1 + x_2) + \cdots + (x_1 + x_2 + \cdots + x_n)
\]
\[
= n x_1 + (n-1) x_2 + \cdots + x_n.
\]
If we replace, the "+" by "\rangle" in the above identity, then the truth values satisfies the equivalence relation.

**References**


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